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Exact 4-point Scattering Amplitude of the Superconformal Schrödinger Chern-Simons Theory

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ABSTRACT

We consider the non-relativistic superconformal $U(N)\times U(N)$ Chern-Simons theory with level $(k,-k)$ possessing fourteen supersymmetries. We obtain an exact four-point scattering amplitude of the theory to all orders in $1/N$ and $1/k$ and prove that the scattering amplitude becomes trivial when $k=1$ and 2. We confirm this amplitude to one-loop order by using an explicit field theoretic computation and show that the beta function for the contact interaction vanishes to the one-loop order, which is consistent with the quantum conformal invariance of the underlying theory.

1 Introduction

Recently there has been much interest in the non-relativistic version of the AdS/CFT correspondence, which has a potential application in some condensed matter systems in the strongly coupled region [1]. Some candidates for such non-relativistic conformal field theories can be obtained by taking appropriate non-relativistic limits of the mass-deformed Aharony-Bergman-Jafferis-Maldacena (ABJM) theory. The ABJM theory is the three dimensional $\mathcal{N}=6$ $U(N)\times U(N)$ superconformal Chern-Simons theory with level $(k,-k)$ and dual to the type IIA string theory on the $AdS_4\times\mathbb{CP}_3$ background [2]. Some tests of this duality have been carried out largely based on the integrability technique [3].

In this note, we shall study the non-relativistic version of the ABJM theory with fourteen supersymmetries [4, 5]. This follows from the mass-deformed ABJM theory [6, 7] by taking the so-called 'PPPP' non-relativistic limit where one chooses all Schrödinger fields from a particle sector instead of an anti-particle sector [4, 5]. The resulting theory possesses $\mathfrak{psu}(2|2)$ symmetry (involving eight kinematical supercharges), together with two extra kinematical and four conformal supersymmetries, leading to fourteen supersymmetries in total. There is also the so-called 'PAAP' non-relativistic limit [4] where one chooses half of the Schrödinger fields from the anti-particle sector in an appropriate manner. This theory has only eight kinematical supersymmetries of the $\mathfrak{psu}(2|2)$. There are non-relativistic limits leading to even less supersymmetric theories [4]. For the other related aspects of non-relativistic supersymmetric Chern-Simons theories, see Refs. [8, 9, 10].

In this note, we focus on the non-relativistic superconformal $U(N)\times U(N)$ Chern-Simons theory with level $(k,-k)$, which has fourteen supersymmetries, and compute its four-point scattering amplitudes describing $2\to 2$ scattering processes. Adopting the previously developed method [11], we derive the two-body Schrödinger equation by which any $2\to 2$ scattering processes can be described. Starting from its scattering solution, we shall obtain an exact four-point scattering amplitude of any combination of incoming two-particle states to all orders in $1/N$ and $1/k$. Because we are dealing with a Chern-Simons gauge theory, there is the so-called statistics interaction, which corresponds to the Aharonov-Bohm interaction (between anyonic particles) characterized by a phase $e^{i\pi\Omega}$, when two particles are exchanged. (The statistics interaction matrix Ω will be specified below.) In addition, there is a two-body contact interaction between particles that can be fully specified by a contact interaction matrix C. These two interaction matrices, in fact, encode the complete interaction structure of our non-relativistic Chern-Simons theory. Any scattering amplitudes can be represented as functions of these two matrices.

We check this four-point amplitude perturbatively by using a direct field theoretic computation to one-loop order. We shall show that the beta function for the contact interaction vanishes to the one-loop order, which is consistent with the quantum conformal symmetry of the underly-

ing theory. After renormalization, the resulting amplitude to the one-loop order agrees precisely with our exact amplitude.

In Section 2, we introduce the non-relativistic ABJM theory with fourteen supersymmetries and discuss the detailed structure of $\mathfrak{psu}(2|2)$ symmetry. Especially, we write the contact and the statistics interaction matrices in manifestly $\mathfrak{psu}(2|2)$ covariant forms. In Section 3, we shall derive the two-body Schrödinger equation that describes generic $2 \to 2$ scattering processes. Section 4 deals with the scattering solution of the two-body Schrödinger equation. Using this solution, we shall extract the exact four-point scattering amplitude. We shall show that the scattering amplitude becomes completely trivial when $k = 1$ and 2. In Section 5, we perform a field theoretic perturbative analysis to the one-loop order. We shall show that the beta function for the contact interaction vanishes for our form of the contact interaction as dictated by the quantum conformal invariance. We shall also show that the perturbative result agrees precisely with the exact amplitude. The last section is devoted to concluding remarks. Some details of the non-relativistic ABJM theory with fourteen supersymmetries are presented in the Appendix.

2 $\mathfrak{psu}(2|2)$ invariance

We shall begin with the non-relativistic Chern-Simons Lagrangian given by

$$\begin{aligned}\mathcal{L} &= \frac{k}{4\pi}CS(A) - \frac{k}{4\pi}CS(\overline{A}) + \mathrm{Tr}\ \Phi_A^\dagger\Big(iD_0 + \frac{1}{2m}\vec{D}^2\Big)\Phi_A \\ &- \frac{1}{4}(\Phi_{\mathcal{A}_1})^\dagger(\Phi_{\mathcal{A}_2})^\dagger C^{\mathcal{A}_1\mathcal{A}_2}_{\mathcal{B}_1\mathcal{B}_2}\,\Phi_{\mathcal{B}_1}\,\Phi_{\mathcal{B}_2}\,, \end{aligned} \tag{2.1}$$

whose detailed component form is presented in Appendix A. Apart from the two gauge fields, A_μ and $\overline{A}_\mu$, there are 8 kinds of complex matter fields, $\Phi_I = (\phi^a\,|\,\psi^\alpha)$ and $\Phi_{\tilde{I}} = (\tilde{\psi}^a\,|\,\tilde{\phi}^\alpha)$ where one has 4 bosons $\phi^a, \tilde{\phi}^\alpha$ and 4 fermions $\psi^\alpha, \tilde{\psi}^a$. We use indices $I,\ J$ for $(\phi^a\,|\,\psi^\alpha)$, $\tilde{I},\ \tilde{J}$ for $(\tilde{\psi}^a\,|\,\tilde{\phi}^\alpha)$ and $A,\ B$ for the total eight flavors. The calligraphic upper case letters run over this 8 flavored matrix component space whose total dimension is $8N^2$. In other words, the index $\mathcal{A}$ represents (Amn); *i.e.*, $\Phi_{\mathcal{A}} = \Phi_{A\,mn}$. The contact interaction matrix C acts upon the two-body state $|\mathcal{B}_1\mathcal{B}_2\rangle$ in the space of $\Phi_1 \otimes \Phi_2$, which ends up with a new two-body state $|\mathcal{A}_1\mathcal{A}_2\rangle$, which is a $64N^2 \times 64N^2$ matrix, whose detailed form will be specified below. This contact interaction matrix will serve as a basic building block of our 4-point scattering amplitude.

Before specifying a detailed form of C, let us first state the $\mathfrak{psu}(2|2)$ invariance of the above Lagrangian. The off-shell $\mathfrak{psu}(2|2)$ superalgebra is spanned by the two su(2) rotation generators $\mathfrak{R}^a{}_b$, $\mathfrak{L}^\alpha{}_\beta$, the supersymmetry generator $\mathfrak{Q}^\alpha_a$ and the superconformal generator $\mathfrak{S}^a_\alpha$. The off-shell configuration is characterized by $sl(2,\mathbb{R})$ central charges $\mathfrak{C}, \mathfrak{K}, \mathfrak{K}^*$. Their commutators are given

by

$$
\begin{aligned}
&[\mathfrak{R}^a{}_b,\, \mathfrak{J}^c] = \delta^c_b \mathfrak{J}^a - \frac{1}{2}\delta^a_b \mathfrak{J}^c\,, \quad [\mathfrak{L}^\alpha{}_\beta,\, \mathfrak{J}^\gamma] = \delta^\gamma_\beta \mathfrak{J}^\alpha - \frac{1}{2}\delta^\alpha_\beta \mathfrak{J}^\gamma\,,\\
&\{\mathfrak{Q}^\alpha_a,\, \mathfrak{S}^b_\beta\} = \delta^b_a \mathfrak{L}^\alpha{}_\beta + \delta^\alpha_\beta \mathfrak{R}^b{}_a + \delta^b_a \delta^\alpha_\beta \mathfrak{C}\,,\\
&\{\mathfrak{Q}^\alpha_a,\, \mathfrak{Q}^\beta_b\} = \varepsilon^{\alpha\beta}\varepsilon_{ab}\mathfrak{K}\,, \quad \{\mathfrak{S}^a_\alpha,\, \mathfrak{S}^b_\beta\} = \varepsilon_{\alpha\beta}\varepsilon^{ab}\mathfrak{K}^*\,.
\end{aligned}
\tag{2.2}
$$

On a state of fundamental representations, the generators act as

$$
\mathfrak{R}^a{}_b|\phi^c\rangle = \delta^c_b|\phi^a\rangle - \frac{1}{2}\delta^a_b|\phi^c\rangle\,, \quad \mathfrak{L}^\alpha{}_\beta|\phi^\gamma\rangle = \delta^\gamma_\beta|\phi^\alpha\rangle - \frac{1}{2}\delta^\alpha_\beta|\phi^\gamma\rangle \tag{2.3}
$$

and as

$$
\begin{aligned}
\mathfrak{Q}^\alpha_a|\phi^b\rangle &= a\,\delta^b_a|\psi^\alpha\rangle\,,\\
\mathfrak{Q}^\alpha_a|\psi^\beta\rangle &= b\,\varepsilon^{\alpha\beta}\varepsilon_{ab}|\phi^b\rangle\,,\\
\mathfrak{S}^a_\alpha|\phi^b\rangle &= c\,\varepsilon_{\alpha\beta}\varepsilon^{ab}|\psi^\beta\rangle\,,\\
\mathfrak{S}^a_\alpha|\psi^\beta\rangle &= d\,\delta^\beta_\alpha|\phi^a\rangle\,.
\end{aligned}
\tag{2.4}
$$

Closure of the superalgebra on a fundamental representation leads to the shortening condition

$$
ad - bc = 1\,. \tag{2.5}
$$

From the algebra, one finds that $\mathfrak{K} = ab$, $\mathfrak{K}^* = cd$ and $\mathfrak{C} = \frac{1}{2}(ad+bc)$. Unitarity of the representation requires that $(\mathfrak{Q}^\alpha_a)^\dagger = \mathfrak{S}^a_\alpha$ and $ab = (cd)^*$.

We have one more set of the fundamental representation where the set $\Phi_I = (\phi^a\,|\,\psi^\alpha)$ in the above is replaced by $\Phi_{\tilde{I}} = (\tilde{\psi}^a\,|\,\tilde{\phi}^\alpha)$ with a further replacement of (a,b,c,d) by $(\tilde{a},\tilde{b},\tilde{c},\tilde{d})$. For Φ_I, we have $a = d = 1$ and $b = c = 0$ while $\tilde{a} = \tilde{d} = -1$ and $\tilde{b} = \tilde{c} = 0$ for $\Phi_{\tilde{I}}$.

To show the $\mathfrak{psu}(2|2)$ invariance of the Lagrangian, let us begin with the contact interaction part. For the specification of the contact interaction terms, we rearrange the quartic potential term in Eq. (A.6) to the normal ordered form $\Phi^\dagger\,\Phi^\dagger\,\Phi\,\Phi$ and use the equal time (anti-) commutation relations

$$
[\Phi_{\mathcal{A}}(\mathbf{r},t)\,,\Phi_{\mathcal{B}}(\mathbf{r}',t)]_\pm \equiv \Phi_{\mathcal{A}}(\mathbf{r},t)\,\Phi_{\mathcal{B}}(\mathbf{r}',t) - (-)^{F_{\mathcal{A}}F_{\mathcal{B}}}\,\Phi_{\mathcal{B}}(\mathbf{r}',t)\,\Phi_{\mathcal{A}}(\mathbf{r},t) = 0\,, \tag{2.6}
$$

where $F_{\mathcal{A}}$ denotes the fermion number of the field $\Phi_{\mathcal{A}}$.

The two-body interaction involving only Φ_I fields is govern by the matrix

$$
S = \mathbb{S} \otimes (\mathrm{P} - \overline{\mathrm{P}})\,, \tag{2.7}
$$

which has only nonvanishing matrix components in $\langle I_1 m_1 n_1, I_2 m_2 n_2|\,S\,|J_1 p_1 q_1, J_2 p_2 q_2\rangle$. The permutations P and $\overline{P}$ are, respectively, defined by

$$
\mathrm{P}\,{}^{m_1 n_1 m_2 n_2}_{p_1 q_1 p_2 q_2} = \delta^{m_2}_{p_1}\delta^{m_1}_{p_2}\delta^{n_1}_{q_1}\delta^{n_2}_{q_2}\,, \qquad \overline{\mathrm{P}}\,{}^{m_1 n_1 m_2 n_2}_{p_1 q_1 p_2 q_2} = \delta^{m_1}_{p_1}\delta^{m_2}_{p_2}\delta^{n_2}_{q_1}\delta^{n_1}_{q_2}\,. \tag{2.8}
$$

The matrix $\mathbb{S}$, which has nonvanishing components only in $\mathbb{S}^{I_1 I_2}_{J_1 J_2}$, can be identified as

$$\mathbb{S}^{I_1 I_2}_{J_1 J_2} = -\frac{2\pi}{mk}\left(\mathbb{I}^{I_1 I_2}_{J_1 J_2} - \mathbb{P}^{I_1 I_2}_{J_1 J_2}\right), \tag{2.9}$$

where the identity $\mathbb{I}$ and the graded permutation $\mathbb{P}$ are, respectively, defined by

$$\mathbb{I}^{B_1 B_2}_{A_1 A_2} = \delta^{B_1}_{A_1}\delta^{B_2}_{A_2}, \qquad \mathbb{P}^{B_1 B_2}_{A_1 A_2} = (-)^{F_{A_1} F_{A_2}}\delta^{B_2}_{A_1}\delta^{B_1}_{A_2}. \tag{2.10}$$

The graded permutation is $\mathfrak{psu}(2|2)$ invariant. Therefore, $\mathbb{S}$ or S are $\mathfrak{psu}(2|2)$ invariant. Similarly, for the contact interaction involving only $\Phi_{\tilde{I}}$ fields, one has an interaction of the form

$$\widetilde{S} = \widetilde{\mathbb{S}} \otimes (\mathrm{P} - \overline{\mathrm{P}}), \tag{2.11}$$

where $\widetilde{\mathbb{S}}$ has only the nonvanishing components

$$\widetilde{\mathbb{S}}^{\tilde{I}_1 \tilde{I}_2}_{\tilde{J}_1 \tilde{J}_2} = +\frac{2\pi}{mk}\left(\mathbb{I}^{\tilde{I}_1 \tilde{I}_2}_{\tilde{J}_1 \tilde{J}_2} - \mathbb{P}^{\tilde{I}_1 \tilde{I}_2}_{\tilde{J}_1 \tilde{J}_2}\right). \tag{2.12}$$

For the interaction involving Φ_I and $\Phi_{\tilde{I}}$ at the same time, we introduce $\mathbb{T}$ defined by

$$\begin{aligned}
&\frac{mk}{2\pi}\mathbb{T}\,|\phi^a\tilde{\phi}^\beta\rangle = -|\tilde{\psi}^a\psi^\beta\rangle - |\psi^\beta\tilde{\psi}^a\rangle, \quad &&\frac{mk}{2\pi}\mathbb{T}\,|\tilde{\phi}^\beta\phi^a\rangle = |\psi^\beta\tilde{\psi}^a\rangle + |\tilde{\psi}^a\psi^\beta\rangle, \\
&\frac{mk}{2\pi}\mathbb{T}\,|\psi^\alpha\tilde{\psi}^b\rangle = |\tilde{\phi}^\alpha\phi^b\rangle - |\phi^b\tilde{\phi}^\alpha\rangle, \quad &&\frac{mk}{2\pi}\mathbb{T}\,|\tilde{\psi}^b\psi^\alpha\rangle = |\tilde{\phi}^\alpha\phi^b\rangle - |\phi^b\tilde{\phi}^\alpha\rangle, \\
&\frac{mk}{2\pi}\mathbb{T}\,|\phi^a\tilde{\psi}^b\rangle = -|\tilde{\psi}^a\phi^b\rangle + |\phi^b\tilde{\psi}^a\rangle, \quad &&\frac{mk}{2\pi}\mathbb{T}\,|\tilde{\psi}^b\phi^a\rangle = -|\phi^b\tilde{\psi}^a\rangle + |\tilde{\psi}^a\phi^b\rangle, \\
&\frac{mk}{2\pi}\mathbb{T}\,|\tilde{\phi}^\alpha\psi^\beta\rangle = |\psi^\alpha\tilde{\phi}^\beta\rangle - |\tilde{\phi}^\beta\psi^\alpha\rangle, \quad &&\frac{mk}{2\pi}\mathbb{T}\,|\psi^\beta\tilde{\phi}^\alpha\rangle = |\tilde{\phi}^\beta\psi^\alpha\rangle - |\psi^\alpha\tilde{\phi}^\beta\rangle.
\end{aligned} \tag{2.13}$$

This matrix is also $\mathfrak{psu}(2|2)$ invariant. In summary, the contact interaction matrix is given by

$$C = (\mathbb{S} + \widetilde{\mathbb{S}} + \mathbb{T}) \otimes (\mathrm{P} - \overline{\mathrm{P}}). \tag{2.14}$$

An interesting property we shall use later on is

$$(\mathbb{S} + \widetilde{\mathbb{S}} + \mathbb{T})^2 = \left(\frac{4\pi}{mk}\right)^2 \frac{\mathbb{I} - \mathbb{P}}{2}. \tag{2.15}$$

This construction ensures the $\mathfrak{psu}(2|2)$ invariance of the contact interaction term in Eq. (2.1).

With $\delta A_\mu = 0$ under the $\mathfrak{psu}(2|2)$ transformation, the kinetic terms are also invariant under the $\mathfrak{psu}(2|2)$ transformation. Thus, we conclude that the system in Eq. (2.1) has $\mathfrak{psu}(2|2)$ invariance. Finally, the tree-level gauge interaction is characterised by the so-called statistical interaction matrix given by

$$\Omega = \frac{1}{k}\,\mathbb{I} \otimes (\mathrm{P} - \overline{\mathrm{P}}). \tag{2.16}$$

We note that its strength is governed by the inverse of the Chern-Simons level and that flavors are not changed by this gauge interaction.

3 Two-body Schrödinger equation

For the Schrödinger Chern-Simons system, the Gauss law constraints

$$F_{12}(A) = \rho_A \quad \text{and} \quad F_{12}(\overline{A}) = \rho_{\overline{A}} \tag{3.1}$$

can be solved explicitly in the gauge $A_1 = \overline{A}_1 = 0$ [11]. One may then eliminate the gauge fields by the solution,

$$A_2 = \frac{1}{\partial_1}\,\rho_A \quad \text{and} \quad \overline{A}_2 = \frac{1}{\partial_1}\,\rho_{\overline{A}}, \tag{3.2}$$

by which one gets an equivalent system that only depends on the matter fields. One may show that no nontrivial functional Jacobian factor arises in this procedure [11]. The resulting action is simply given by

$$\mathcal{L} = \mathrm{Tr}\;\Phi_A^\dagger\Big(i\partial_0 + \frac{1}{2m}\vec{D}^2\Big)\,\Phi_A - \frac{1}{4}\,(\Phi_{\mathcal{A}_1})^\dagger(\Phi_{\mathcal{A}_2})^\dagger\, C\,{}^{\mathcal{A}_1\mathcal{A}_2}_{\mathcal{B}_1\mathcal{B}_2}\;\Phi_{\mathcal{B}_1}\,\Phi_{\mathcal{B}_2}\,, \tag{3.3}$$

where the covariant derivative is defined with the solution in Eq. (3.2).

The n-body Schrödinger equation can be derived using the operator Schrödinger equation following a straightforward procedure [11]. The $2 \to 2$ scattering amplitude of our interest can be obtained using this two-body Schrödinger equation. The two-body wave function is defined by

$$\Psi_{\mathcal{A}_1\mathcal{A}_2}(\mathbf{r}_1,\mathbf{r}_2;t) = \langle 0|\Phi_{\mathcal{A}_1}(\mathbf{r}_1,t)\Phi_{\mathcal{A}_2}(\mathbf{r}_2,t)|\Psi\rangle = \langle 0|\Phi_{A_1\,m_1n_1}(\mathbf{r}_1,t)\Phi_{A_2\,m_2n_2}(\mathbf{r}_2,t)|\Psi\rangle\,, \tag{3.4}$$

which has the exchange symmetry

$$\Psi_{\mathcal{A}_1\mathcal{A}_2}(\mathbf{r}_1,\mathbf{r}_2;t) = (-)^{F_{\mathcal{A}_1}F_{\mathcal{A}_2}}\Psi_{\mathcal{A}_2\mathcal{A}_1}(\mathbf{r}_2,\mathbf{r}_1;t) = (\mathbb{P}\,\mathrm{P}\,\overline{\mathrm{P}}\,\Psi)_{\mathcal{A}_1\mathcal{A}_2}(\mathbf{r}_2,\mathbf{r}_1;t)\,. \tag{3.5}$$

We find that the two-body Schrödinger equation takes the following form:

$$i\partial_t\,\Psi(\mathbf{r}_1,\mathbf{r}_2;t) = \Big[-\frac{1}{2m}(\nabla_1 + 2\pi i\Omega\,\mathbf{G}(\mathbf{r}_{12})\,)^2 + (1\leftrightarrow 2) + \frac{C}{2}\,\delta(\mathbf{r}_{12})\Big]\Psi(\mathbf{r}_1,\mathbf{r}_2;t)\,, \tag{3.6}$$

where $G_i(\mathbf{r}) = \frac{1}{2\pi}\varepsilon_{ij}\partial_j \ln r$ and $\mathbf{r}_{12} = \mathbf{r}_1 - \mathbf{r}_2$. One thing to note is that we have performed a non-singular gauge transformation from the axial gauge to the Coulomb gauge $\nabla\cdot\mathbf{G} = 0$. In the center-of-momentum frame, the time-independent Schrödinger equation becomes

$$\Big[-\frac{1}{m}(\nabla + 2\pi i\,\Omega\,\mathbf{G}(\mathbf{r})\,)^2 + \frac{C}{2}\,\delta(\mathbf{r}) - E\Big]\Psi(\mathbf{r}) = 0\,, \tag{3.7}$$

where, with relative momentum $\mathbf{p}$, we take $E = \frac{p^2}{m}$ for our scattering problem. This equation will be the starting point of our analysis in the next section for the construction of our scattering amplitude.

4 Exact $2 \to 2$ scattering amplitude

Before solving our main problem for the full scattering amplitude, let us first consider the single-component Schrödinger equation given by

$$\Big[-\frac{1}{m}(\nabla+2\pi i\nu\,\mathbf{G}(\mathbf{r}))^2+\frac{c}{2}\delta(\mathbf{r})-\frac{p^2}{m}\Big]\psi(\mathbf{r})=0\,, \tag{4.1}$$

where ν and c are not matrix-valued, but c-numbers. This may be viewed as an eigen-component equation in the basis where Ω and C are simultaneously diagonalized.

The scattering solution with the boundary condition $\psi(0)=0$ reads [12, 13, 14, 15]

$$\psi(r,\theta)=e^{ipr\cos\theta-i\nu(\bar{\theta}-\pi)}-\sin\nu\pi\, e^{-i([\nu]+1)\theta}\int_{-\infty}^{\infty}\frac{dt}{\pi}e^{ipr\cosh t}\frac{e^{-\{\nu\}t}}{e^{-i\theta}-e^{-t}}\,, \tag{4.2}$$

where $[\nu]$ denotes the greatest integer part of ν, $\{\nu\}=\nu-[\nu]$ and $\bar{\theta}=\theta-2\pi n$ if $2\pi n\le\theta<2\pi(n+1)$ $(n\in\mathbb{Z})$. This definition of $\bar{\theta}$ ensures the single-valuedness of the above wave function [15]. Note also that we have not imposed the exchange symmetry property of the wave function yet. In order to get the above solution, one begins with the partial wave analysis, where one takes $e^{ipr\cos\theta}$ as an initial wave function. The resulting partial wave solution with the $\psi(0)=0$ boundary condition can be summed, leading to the above integral form.

Later, we shall show that the eigenvalues of Ω range over $-2\le\nu\le 2$. When $1\le|\nu|\le 2$, solely the cases of $\nu=\pm1,\pm2$ are relevant for our application below. We shall treat these four cases separately. The remaining possibility then lies only within the interval $(-1,1)$. For this range of ν, one may consider the self-adjoint extension of the s-wave (zero orbital angular momentum) part, which is relevant for a general understanding of the scattering amplitude. The extension is dictated by the boundary condition

$$\Big[r^{|\nu|}\,\psi(r)-R^{2|\nu|}\,\frac{d\,r^{|\nu|}\psi(r)}{d\,r^{2|\nu|}}\Big]_{r=0}=0\,, \tag{4.3}$$

where R is a scale-dependent parameter related to the RG-scale of field theory [16].

In our case, we also require quantum scale invariance, which follows from the superconformal invariance of the underlying theory. This selects only two possibilities. One is the $R=0$ case or equivalently $\psi(0)=0$, which is called the "repulsive critical" boundary condition. The other scale-invariant boundary condition is $\frac{d\,r^{|\nu|}\psi(r)}{d\,r^{2|\nu|}}\,|_{r=0}=0$, which we call the "attractive critical" boundary condition. For the former boundary condition, the scattering solution was already presented in Eq. (4.2). For the latter, the scattering solution becomes

$$\begin{aligned}\psi(r,\theta)\;&=\;e^{ipr\cos\theta-i\nu(\bar{\theta}-\pi)}-\sin\nu\pi\, e^{-i([\nu]+1)\theta}\int_{-\infty}^{\infty}\frac{dt}{\pi}e^{ipr\cosh t}\frac{e^{-\{\nu\}t}}{e^{-i\theta}-e^{-t}}\\&-\;\;[e^{-\frac{|\nu|\pi i}{2}}J_{|\nu|}(pr)-e^{\frac{|\nu|\pi i}{2}}J_{-|\nu|}(pr)]\,.\end{aligned} \tag{4.4}$$

For $\nu = \pm 1, \pm 2$, the self-adjoint extension is not allowed, and the scattering solution becomes almost trivial:

$$\psi(r,\theta) \;=\; e^{ipr\cos\theta - i\nu(\theta-\pi)}\,. \tag{4.5}$$

The scattering amplitude is defined by the asymptotic form

$$\psi \sim e^{ipr\cos\theta} + \frac{1}{\sqrt{r}}\, e^{i(pr+\frac{\pi}{4})}\, f(\theta) \tag{4.6}$$

in the large-r limit. For $|\nu| < 1$, the scattering amplitude can be found as

$$f_{\pm}(\theta) = f_{\rm ns}(\theta) + f_{{\rm s}\pm}(\theta)\,, \tag{4.7}$$

where the non-s-wave and the s-wave contributions, $f_{\rm ns}(\theta)$ and $f_{\rm s}(\theta)$, are, respectively, given by

$$f_{\rm ns}(\theta) \;=\; -\frac{i}{\sqrt{2\pi p}}\left[\sin\pi\nu\,\cot\frac{\theta}{2} + 2\sin^2\frac{\pi\nu}{2}\,(1 - 2\pi\delta(\theta))\right] \tag{4.8}$$

$$f_{{\rm s}\pm}(\theta) \;=\; -\frac{i}{\sqrt{2\pi p}}\,(\,e^{\mp i\pi|\nu|} - 1\,)\,. \tag{4.9}$$

Here, the upper and the lower signs are, respectively, for the repulsive and the attractive critical boundary conditions. The term involving the forward delta function arises from the phase-modulated incoming wave in Eqs. (4.2), (4.4) and (4.5). Its scattering part has the asymptotic form

$$(e^{-i\nu(\theta-\pi)} - 1)\, e^{ipr\cos\theta} \to (e^{-i\nu(\theta-\pi)} - 1)\,\frac{1}{\sqrt{2\pi pr}}\, e^{i(pr-\frac{\pi}{4})} \sum_{n=-\infty}^{\infty} e^{in\theta}$$

$$= (e^{-i\nu(\theta-\pi)} - 1)\frac{e^{i(pr-\frac{\pi}{4})}}{\sqrt{2\pi pr}}\, 2\pi\delta(\theta) = (\cos\nu\pi - 1)\frac{e^{i(pr-\frac{\pi}{4})}}{\sqrt{2\pi pr}}\, 2\pi\delta(\theta)\,, \tag{4.10}$$

where, for the last equality, we average the $\theta = 0$ and the $\theta = 2\pi$ contributions. Though we shall not present any further details, the presence of the delta function can also be checked from the unitarity requirement of the scattering matrix. For the $\nu = \pm 1, \pm 2$ cases, the scattering amplitude directly obtained from Eq. (4.5) agrees with the expression in Eq. (4.9) if the corresponding values are substituted.

Let us now discuss the scattering amplitude from the view point of a field theory perturbative analysis. We first note that the strength of the contact interaction is not arbitrary. The quantum scale-invariance requires

$$\left(\frac{mc}{4}\right)^2 = (\pi\nu)^2\,, \tag{4.11}$$

on which the β-function for the interaction strength c vanishes. This aspect will be discussed in detail later, but we assume that c takes values that satisfy this condition of criticality.

From the perturbative analysis, one may show that the non s-wave part does not arise at all from amplitudes that contain more than one contact interaction vertex. Then, using the equivalence between quantum mechanics and the non-relativistic field theory, we claim that the non-s-wave field theory amplitude should be given by $f_{\rm ns}(\theta)$ in Eq. (4.8).

On the other hand, the s-wave part receives contributions involving an odd number of the contact interaction vertices. There can be a dependence on even numbers of c, which can be replaced by the same powers of ν. One may also show that the s-wave part involves only even powers of ν without any odd powers of ν. Furthermore, the perturbative result obviously requires that $f_{\rm s}$ should be analytic in c and ν. Hence, using Eq. (4.11) and the equivalence between the quantum mechanics and the non-relativistic field theory, the field theoretic *critical* s-wave amplitude should be

$$f_{\rm s} = -\frac{i}{\sqrt{2\pi p}}\left[-i\sin(\frac{mc}{4}) + \cos\pi\nu - 1\right]. \tag{4.12}$$

One finally imposes the exchange symmetry

$$f^{\rm full}(\theta) = f(\theta) + (-)^{F_1F_2} f(\theta+\pi), \tag{4.13}$$

where F_1 and F_2 are fermion numbers of incoming particles.

This result can be extended even for general boundary conditions, including scale-variant values of c, which was first suggested in Ref. [16] based on the two-loop analysis and was proven to all-loop orders in Ref. [17]. A similar analysis can be carried out for our problem starting from the Schrödinger equation in Eq. (3.7). One may start from the scattering solution of the Schrödinger equation in Eq. (3.7), which is given by the solution in Eq. (4.2) with the replacement of ν by the matrix Ω. In this solution, one is defining $|\Omega|$ in the basis where Ω is diagonal, which is achieved by diagonalizing $\mathrm{P}-\overline{\mathrm{P}}$. Its eigenspaces are given by the projections

$$\Pi_0^g = \frac{1}{2}(1+\mathrm{P}\overline{\mathrm{P}}), \quad \Pi_\pm^g = \frac{1}{4}\Big(2\pm(\mathrm{P}-\overline{\mathrm{P}}\Big)\,\frac{1}{2}(1-\mathrm{P}\overline{\mathrm{P}}) \tag{4.14}$$

with eigenvalues

$$\Omega\,\Pi_0^g = 0, \quad \Omega\,\Pi_\pm^g = \pm\Big(\frac{2}{k}\Big)\,\Pi_\pm^g. \tag{4.15}$$

The non-s-wave scattering amplitude is simply given by

$$F_{ns}(\theta) = -\frac{i}{\sqrt{2\pi p}}\left[\sin\pi\Omega\cot\frac{\theta}{2} + 2\sin^2\frac{\pi\Omega}{2}\,(1-2\pi\,\delta(\theta))\right]. \tag{4.16}$$

This can be proven by going to the eigenspace where one may use our previous result for the single component. Including the exchange symmetries, one has

$$F_{ns}^{\text{full}}(\theta) = F_{ns}(\theta) + F_{ns}(\theta+\pi)\mathbb{P}\,\mathrm{P}\overline{\mathrm{P}}\,. \tag{4.17}$$

For the s-wave contribution, let us first note a few things. We introduce a new gauge interaction matrix $\widetilde{\Omega}$ as

$$\widetilde{\Omega} = \frac{1}{2k}\,(\mathbb{I}-\mathbb{P})\otimes(\mathrm{P}-\overline{\mathrm{P}})\,, \tag{4.18}$$

which includes the effect of the s-wave exchange symmetry. Namely, the projection

$$\Pi^{\text{ex}} = \frac{1}{2}(1+\mathbb{P}\,\mathrm{P}\overline{\mathrm{P}}) \tag{4.19}$$

defines the projection of in and out s-wave states, and the operator Ω acting upon the s-wave projected state becomes $\widetilde{\Omega}$:

$$\Omega\,\Pi^{\text{ex}} = \widetilde{\Omega}\,\Pi^{\text{ex}} = \Pi^{\text{ex}}\,\widetilde{\Omega}\,. \tag{4.20}$$

As will be shown in detail later on, the quantum scale-invariance of our Chern-Simons theory is maintained by the relation

$$\left(\frac{mC}{4}\right)^2 = (\pi\widetilde{\Omega})^2\,, \tag{4.21}$$

for which one has to consider the full s-wave contribution, which respects the exchange symmetry. The supermatrix relation Eq. (4.21) follows from the property of C in Eq. (2.15).

For the eigenspace of the supermatrix C, we further introduce projections

$$\Pi_0^f = \frac{1}{2}(1+\mathbb{P})\,,\quad \Pi_\pm^f = \frac{1}{2}\Big(1 \pm \frac{mk}{4\pi}(\mathbb{S}+\widetilde{\mathbb{S}}+\mathbb{T})\Big)\,\frac{1}{2}(1-\mathbb{P})\,. \tag{4.22}$$

Then, C and $\widetilde{\Omega}$ are simultaneously diagonalized as

$$\begin{aligned}
&C\,\Pi_0^g = C\,\Pi_0^f = \widetilde{\Omega}\,\Pi_0^g = \widetilde{\Omega}\,\Pi_0^f = 0\,,\\
&C\,\Pi_\pm^f\,\Pi_\pm^g = (\pm)\,(\pm)\frac{8\pi}{mk}\,\Pi_\pm^f\,\Pi_\pm^g\,,\\
&\widetilde{\Omega}\,\Pi_\pm^f\,\Pi_\pm^g = (+)\,(\pm)\frac{2}{k}\,\Pi_\pm^f\,\Pi_\pm^g\,,
\end{aligned} \tag{4.23}$$

where the signatures of the first and the second parentheses are, respectively, those of the first and second projections. Since, in the eigenbasis, the previous analysis in Eq. (4.12) is valid, the full s-wave contribution is given by

$$F_{\text{s}}^{\text{full}} = -\frac{2i}{\sqrt{2\pi p}}\Big[-i\sin(\frac{mC}{4}) + \cos\pi\widetilde{\Omega} - 1\Big]\,\Pi^{\text{ex}}\,. \tag{4.24}$$

The exact $2 \to 2$ scattering amplitude is then given by the scattering matrix

$$F^{\text{full}} = F_{\text{ns}}^{\text{full}} + F_{\text{s}}^{\text{full}}, \tag{4.25}$$

whose component $\langle \mathcal{A}'_1 \mathcal{A}'_2 \,|\, F^{\text{full}} \,|\, \mathcal{A}_1 \mathcal{A}_2 \rangle$ describes the $\mathcal{A}_1 \mathcal{A}_2 \to \mathcal{A}'_1 \mathcal{A}'_2$ scattering amplitude in the component notation.

Using this exact construction, one finds that the $k = 1, 2$ cases are special. For $k = 1$, the scattering amplitude vanishes completely. For $k = 2$, the scattering amplitude F^{full} involves only forward and backward delta-function scattering contributions, which are, respectively, proportional to $\delta(\theta)$ and $\delta(\theta + \pi)$. These contributions proportional to delta functions can be absorbed into the definition of incoming waves, which leads to null scattering. This implies that they are simply artifacts of our scattering formulation. Therefore, we conclude that 4-point scattering is completely trivial for the special cases of $k = 1$ and 2. This observation is also consistent with the supersymmetry enhancement observed in Ref. [8].

5 Perturbative amplitude and β-function

In this section, we shall present checks of our exact scattering amplitude by using a direct field theory computation. Basically, the same perturbative analysis with an arbitrary gauge group, but with a purely bosonic matter field appears in Ref. [15], so we do not need any new computation of Feynman diagrams. We need to simply include the statistical factors arising whenever two fermi fields are exchanged in the computation of Wick contractions.

Adopting the convention in Ref. [15], we shall use the Coulomb gauge, adding the terms

$$\mathcal{L}_{gf} = \frac{1}{\xi} \mathrm{Tr}\, (\nabla \cdot \mathbf{A})^2 + \frac{1}{\xi} \mathrm{Tr}\, (\nabla \cdot \bar{\mathbf{A}})^2 \tag{5.1}$$

to the Lagrangian in Eq. (2.1), and take the limit, $\xi \to 0$ and $\bar{\xi} \to 0$. The corresponding ghost terms

$$\mathcal{L}_{gh} = \mathrm{Tr}\, \eta^\dagger \Big(\nabla^2 \eta + i\, [A_i\,, \partial_i \eta\,] \Big) + \mathrm{Tr}\, \bar{\eta}^\dagger \Big(\nabla^2 \bar{\eta} + i\, [\bar{A}_i\,, \partial_i \bar{\eta}\,] \Big) \tag{5.2}$$

should also be included. Below, we shall use the dimensional regularization with spatial dimension $d = 2 - 2\varepsilon$, plus one dimension corresponding to the time direction.

Following computation in Ref. [15], one may show that the gluon self-energy contribution vanishes identically to the one-loop order. In addition, propagators for the matter fields do not receive any higher-order corrections. This basically follows from the fact that, in non-relativistic theories, pair creations are not possible at all.

The tree-level 4-point amplitude becomes

$$A_{(0)} = \Big(\frac{C}{2} - \frac{2\pi i}{m}\,\Omega\cot\frac{\theta}{2}\Big) + \Big(\frac{C}{2} + \frac{2\pi i}{m}\,\Omega\tan\frac{\theta}{2}\Big)\,\mathbb{P}\mathrm{P}\overline{\mathrm{P}}\,, \tag{5.3}$$

which is consistent with our exact scattering amplitude of the previous section. For the one-loop order, the regularized amplitude may be evaluated as

$$\begin{aligned} A_{(1)} &= \frac{m}{8\pi}\Big(C^2 - \frac{16\pi^2}{m^2}\,\widetilde{\Omega}^2\Big)\Big(\frac{1}{\varepsilon} + \ln\big(\frac{4\pi\mu^2}{p^2}\big) + i\pi - \gamma\Big) \\ &+ \frac{2\pi^2 i}{m}\Big(\delta(\theta)\,\Omega^2 + \delta(\theta+\pi)\,\Omega^2\,\mathbb{P}\mathrm{P}\overline{\mathrm{P}}\Big)\,, \end{aligned} \tag{5.4}$$

where γ is Euler's constant. The amplitude is renormalized by adding a counter-term corresponding to

$$C = C_{\mathrm{ren}} + \delta C\,, \tag{5.5}$$

with

$$\delta C = -\frac{m}{8\pi}\Big(C_{\mathrm{ren}}^2 - \frac{16\pi^2}{m^2}\,\widetilde{\Omega}^2\Big)\Big(\frac{1}{\varepsilon} + \ln 4\pi - \gamma\Big)\,. \tag{5.6}$$

The renormalized amplitude becomes

$$A_{(0)}^{\mathrm{ren}} = \Big(\frac{C_{\mathrm{ren}}}{2} - \frac{2\pi i}{m}\,\Omega\cot\frac{\theta}{2}\Big) + \Big(\frac{C_{\mathrm{ren}}}{2} + \frac{2\pi i}{m}\,\Omega\tan\frac{\theta}{2}\Big)\,\mathbb{P}\mathrm{P}\overline{\mathrm{P}} \tag{5.7}$$

and

$$A_{(1)}^{\mathrm{ren}} = \frac{m}{8\pi}\Big(C_{\mathrm{ren}}^2 - \frac{16\pi^2}{m^2}\,\widetilde{\Omega}^2\Big)\Big(\ln\frac{\mu^2}{p^2} + i\pi\Big) + \frac{2\pi^2 i}{m}\Big(\delta(\theta)\,\Omega^2 + \delta(\theta+\pi)\,\Omega^2\,\mathbb{P}\mathrm{P}\overline{\mathrm{P}}\Big)\,. \tag{5.8}$$

The one-loop β function for the contact interaction matrix becomes

$$\beta_C \equiv \frac{dC_{\mathrm{ren}}}{d\ln\mu} = \frac{m}{4\pi}\Big(C_{\mathrm{ren}}^2 - \frac{16\pi^2}{m^2}\,\widetilde{\Omega}^2\Big)\,, \tag{5.9}$$

which becomes critical if

$$C_{\mathrm{ren}}^2 - \frac{16\pi^2}{m^2}\,\widetilde{\Omega}^2 = 0\,. \tag{5.10}$$

This is precisely the relation satisfied by our contact interaction matrix, which is also consistent with the quantum scale-invariance of the Chern-Simons theory. Upon using the relation in Eq. (5.10), one confirms that the amplitude up to one loop agrees precisely with that of our exact analysis. This analysis can be extended to higher orders [16, 17], but we shall not go to that direction.

6 Conclusions

In this note, we consider the non-relativistic superconformal Chern-Simons theory with fourteen supersymmetries. We show that the theory is completely specified by the contact and the statistics interaction matrices, C and Ω, which respect the $\mathfrak{psu}(2|2)$ symmetry of the underlying theory.

Restricted to the two-body sector of the non-relativistic system, we derive the corresponding two-body Schrödinger equation. From its $2 \to 2$ scattering solution, we obtain the exact four-point scattering amplitude valid to all orders in $1/N$ and $1/k$. We show that the scattering amplitude becomes completely trivial when $k = 1$ and 2. We confirm this scattering amplitude to one-loop order by using a field theoretic perturbative computation. Especially, we verify that the beta function for the contact interaction vanishes to one-loop order, which is consistent with the quantum conformal invariance of the theory. Extending this check to two-loop order is straightforward.

It would be interesting to extend our analysis to cases of less supersymmetric theories, some of which can be obtained from the mass-deformed ABJM theories by using appropriate non-relativistic limits. Especially, the 'PAAP' theory involves only kinematical supersymmetries, so an explicit check of quantum conformal invariance by computing its beta function of contact interaction would be quite interesting.

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A Non-relativistic Chern-Simons theory

The maximally supersymmetric non-relativistic Chern-Simons system is described by [4, 5]

$$\begin{aligned}\mathcal{L} &= \frac{k}{4\pi}CS(A) - \frac{k}{4\pi}CS(\overline{A}) \\ &+ \mathrm{Tr}\Big[\,\phi_a^\dagger\Big(iD_0 + \frac{1}{2m}\vec{D}^2\Big)\phi^a + \tilde{\phi}_\alpha^\dagger\Big(iD_0 + \frac{1}{2m}\vec{D}^2\Big)\tilde{\phi}^\alpha\,\Big] \\ &+ \mathrm{Tr}\Big[\psi_\alpha^\dagger\Big(iD_0 + \frac{1}{2m}\vec{D}^2\Big)\psi^\alpha + \tilde{\psi}_a^\dagger\Big(iD_0 + \frac{1}{2m}\vec{D}^2\Big)\tilde{\psi}^a\Big] - V\,, \end{aligned} \tag{A.1}$$

where

$$CS(A) = \varepsilon^{\mu\nu\lambda}\mathrm{Tr}\Big(A_\mu\partial_\nu A_\lambda + \frac{2i}{3}A_\mu A_\nu A_\lambda\Big)\,, \tag{A.2}$$

and

$$
\begin{aligned}
V &= \frac{1}{2m}\mathrm{Tr}\Big[\,(\psi^\alpha\psi_\alpha^\dagger-\tilde\psi^a\tilde\psi_a^\dagger)\,F_{12}(A)+(\psi_\alpha^\dagger\psi^\alpha-\tilde\psi_a^\dagger\tilde\psi^a)\,F_{12}(\overline{A})\,\Big]\\
&+\frac{\pi}{mk}\mathrm{Tr}\Big[\,\phi_a^\dagger\phi^a\,\phi_b^\dagger\phi^b-\phi^a\phi_a^\dagger\,\phi^b\phi_b^\dagger-\tilde\phi_\alpha^\dagger\tilde\phi^\alpha\,\tilde\phi_\beta^\dagger\tilde\phi^\beta+\tilde\phi^\alpha\tilde\phi_\alpha^\dagger\,\tilde\phi^\beta\tilde\phi_\beta^\dagger\,\Big]\\
&+\frac{\pi}{mk}\mathrm{Tr}\Big[(\phi_a^\dagger\phi^a-\tilde\phi_\alpha^\dagger\tilde\phi^\alpha)(\psi_\beta^\dagger\psi^\beta+\tilde\psi_b^\dagger\tilde\psi^b)+(\phi^a\phi_a^\dagger-\tilde\phi^\alpha\tilde\phi_\alpha^\dagger)(\psi^\beta\psi_\beta^\dagger+\tilde\psi^b\tilde\psi_b^\dagger)\Big]\\
&-\frac{2\pi}{mk}\mathrm{Tr}\Big[\phi_a^\dagger\tilde\psi^a\tilde\psi_b^\dagger\phi^b+\phi^a\tilde\psi_a^\dagger\tilde\psi^b\phi_b^\dagger+\psi_\alpha^\dagger\tilde\phi^\alpha\tilde\phi_\beta^\dagger\psi^\beta+\psi^\alpha\tilde\phi_\alpha^\dagger\tilde\phi^\beta\psi_\beta^\dagger\Big]\\
&-\frac{2\pi}{mk}\mathrm{Tr}\Big[\psi_\alpha^\dagger\tilde\phi^\alpha\tilde\psi_b^\dagger\phi^b+\psi^\alpha\tilde\phi_\alpha^\dagger\tilde\psi^b\phi_b^\dagger+\phi_a^\dagger\tilde\psi^a\tilde\phi_\beta^\dagger\psi^\beta+\phi^a\tilde\psi_a^\dagger\tilde\phi^\beta\psi_\beta^\dagger\Big]\,. \qquad \text{(A.3)}
\end{aligned}
$$

The indices a,b and α,β are, respectively, for the first and the second $\mathrm{SU}(2)$ of $\mathfrak{psu}(2|2)$. All of them run over $\{1,2\}$. The indices μ,ν,λ are for the spacetime directions, which run over $\{0,1,2\}$, and we use vector notation for the spatial directions. All the fields are $N\times N$ matrix valued, and the covariant derivative is defined by

$$
D_\mu\Phi=\partial_\mu\Phi+iA_\mu\Phi-i\Phi\overline{A}_\mu \qquad \text{(A.4)}
$$

such that the system possesses the $\mathrm{U(N)}\times\mathrm{U(N)}$ gauge symmetry. We use the Gauss law constraints

$$
\begin{aligned}
\frac{k}{2\pi}F_{12}(A)&=\rho_A\equiv\phi^a\phi_a^\dagger+\tilde\phi^\alpha\tilde\phi_\alpha^\dagger-\psi^\alpha\psi_\alpha^\dagger-\tilde\psi^a\tilde\psi_a^\dagger\,,\\
\frac{k}{2\pi}F_{12}(\overline{A})&=\rho_{\overline{A}}\equiv\phi_a^\dagger\phi^a+\tilde\phi_\alpha^\dagger\tilde\phi^\alpha+\psi_\alpha^\dagger\psi^\alpha+\tilde\psi_a^\dagger\tilde\psi^a \qquad \text{(A.5)}
\end{aligned}
$$

to eliminate the $F_{12}(A)$ and the $F_{12}(\overline{A})$ terms. The potential then becomes

$$
V=V_{BB}+V_{FF}+V_{BF}+\widetilde{V}_{BF}\,, \qquad \text{(A.6)}
$$

with

$$
\begin{aligned}
V_{BB} &= \frac{\pi}{mk}\mathrm{Tr}\Big[\phi_a^\dagger\phi^a\phi_b^\dagger\phi^b-\phi^a\phi_a^\dagger\phi^b\phi_b^\dagger-\tilde\phi_\alpha^\dagger\tilde\phi^\alpha\tilde\phi_\beta^\dagger\tilde\phi^\beta+\tilde\phi^\alpha\tilde\phi_\alpha^\dagger\tilde\phi^\beta\tilde\phi_\beta^\dagger\Big]\,,\\
V_{FF} &= \frac{\pi}{mk}\mathrm{Tr}\Big[\psi_\alpha^\dagger\psi^\alpha\psi_\beta^\dagger\psi^\beta-\psi^\alpha\psi_\alpha^\dagger\psi^\beta\psi_\beta^\dagger-\tilde\psi_a^\dagger\tilde\psi^a\tilde\psi_b^\dagger\tilde\psi^b+\tilde\psi^a\tilde\psi_a^\dagger\tilde\psi^b\tilde\psi_b^\dagger\Big]\,,\\
V_{BF} &= \frac{2\pi}{mk}\mathrm{Tr}\Big[\phi_a^\dagger\phi^a\psi_\beta^\dagger\psi^\beta+\phi^a\phi_a^\dagger\psi^\beta\psi_\beta^\dagger-\tilde\phi_\alpha^\dagger\tilde\phi^\alpha\tilde\psi_b^\dagger\tilde\psi^b-\tilde\phi^\alpha\tilde\phi_\alpha^\dagger\tilde\psi^b\tilde\psi_b^\dagger\Big]\,,\\
\widetilde{V}_{BF} &= -\frac{2\pi}{mk}\mathrm{Tr}\Big[\phi_a^\dagger\tilde\psi^a\tilde\psi_b^\dagger\phi^b+\phi^a\tilde\psi_a^\dagger\tilde\psi^b\phi_b^\dagger+\psi_\alpha^\dagger\tilde\phi^\alpha\tilde\phi_\beta^\dagger\psi^\beta+\psi^\alpha\tilde\phi_\alpha^\dagger\tilde\phi^\beta\psi_\beta^\dagger\\
&\qquad +\psi_\alpha^\dagger\tilde\phi^\alpha\tilde\psi_b^\dagger\phi^b+\psi^\alpha\tilde\phi_\alpha^\dagger\tilde\psi^b\phi_b^\dagger+\phi_a^\dagger\tilde\psi^a\tilde\phi_\beta^\dagger\psi^\beta+\phi^a\tilde\psi_a^\dagger\tilde\phi^\beta\psi_\beta^\dagger\Big]\,. \qquad \text{(A.7)}
\end{aligned}
$$

This is the form of the potential we use in the main body of this paper.

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